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The state of stress induced by the plane frictionless cylindrical contact. II. The general case (elastic dissimilarity)

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Abstract

In Part I of the paper, the authors have studied the contact problem between a pin and an infinite plate containing a conforming hole, in the absence of friction and in the case of elastic similarity, obtaining a closed form result which generalizes the identical materials analysis of Persson (On the stress distribution of cylindrical elastic bodies in contact, Ph.D. dissertation, 1964).

Here, in Part II, the general case of contacting materials is first studied numerically, finding that the effect of elastic dissimilarity (i.e. the second Dundurs' constant not being zero) is negligible for the dimensionless pressure distribution, the maximum influence being less than 2%. Vice versa, the influence on the relation between the contact area arc semi-width, ε, and the dimensionless loading parameter $E_1^*\Delta R/Q$ is indeed significant; however, considering as an approximate pressure distribution the one of the elastically similar case, an extremely good approximation is obtained for the general relation ε vs. $E_1^*\Delta R/Q$ which can now take into account of both Dundurs' elastic parameters. In particular, the limiting value for $\varepsilon_{\rm lim}$, towards which the contact tends under very high loads both under initial clearance or interference (or for any load for the perfect fit limiting case) is given as a function of both Dundurs' elastic parameters, α, β as well as the load when complete contact is lost in an interference contact, $\varepsilon_{\rm compl}$.

Hence, a complete assessment of the strength of the contact can be obtained directly from the results of Part I of the paper, given that for a certain contact area extension, the correct value of load is used.

1. Introduction

Joints of common use in mechanical assemblies, like rivets or fasteners, can be adequately individually modeled as pin journalled into an infinite plate and transmitting a normal load. Within the frictionless contact hypothesis, the resulting contact problem can be treated analytically, as indeed was done by

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Shtayerman (1949, Section II.7), and by Persson (1964) in his Ph.D. thesis. In particular, the latter gave an *almost* closed form solution to the limiting case of identical contacting materials (*almost* because the relation load to contact area was only given in quadrature). Noble and Hussain (1969) derived independently a completely closed form solution in the more general case of elastic similarity of contacting materials, i.e. for Dundurs' parameter $\beta = 0$. However, their solution was limited to the case of zero clearance (neat-fit condition). This state of affairs moved the present authors to use results from both solutions, developing closed form formulae for both pressure distribution and relation load to contact area size with arbitrary initial gap or interference, with the only assumption of $\beta = 0$ (Part I of the present paper).

Here, in this second part of the paper, the effect of relaxing the assumption $\beta = 0$, is dealt with an efficient numerical method. It is found that the effect of the second Dundurs' parameter, β, is large on the relation contact area to load, but it is very small on the pressure distribution itself, when it has been non-dimensionalized by the load. As a result, if the pressure distribution is assumed to be equal to the case of elastic similarity, the error is less than 2%. Furthermore, not only the relation contact area to load is obtained in closed form, but also the results of Part I of the paper regarding strength of the contact (or maximum of von Mises stress) became of immediate use for the general case.

2. Formulation

Since the geometry of the problem, the nomenclature and the formulation are as in Part I of the paper. However, here we recall Persson's general governing equations

$$\int_{-b}^{+b} \frac{q'(t)}{t-y}\,\mathrm{d}t + \frac{2\pi\beta}{1+y^2} q(y) = -\frac{4(1-\beta)}{(1+y^2)^2} + \frac{B}{1+y^2}, \tag{1}$$

where

$$B = 2(1-\beta) - (1-\alpha)\int_{-b}^{+b} q(t)\frac{\mathrm{d}t}{1+t^2} - \frac{\pi}{2}(1+\alpha)\frac{E_1^*\Delta R}{Q} \tag{2}$$

and the closure condition is given by the following:

$$\int_{-b}^{+b} q(t)\frac{1-t^2}{(1+t^2)^2}\,\mathrm{d}t = \frac{1}{2}. \tag{3}$$

The auxiliary variables are as given in Part I, that is

$$y = \tan(\phi/2), \qquad t = \tan(\psi/2), \qquad b = \tan(\varepsilon/2), \tag{4}$$

where ε is the semi-angle of contact, ϕ and ψ are defined in Fig. 1 of Part I.

3. Numerical solution

The numerical solution of the governing equations (1)–(3) is straightforward: B is substituted in Eq. (4), the resulting singular integral equation (i) is integrated by part in order to substitute the first derivative of $q(y)$ with $q(y)$ itself, giving

$$\int_{-b}^{+b} \frac{q'(t)}{t-y}\mathrm{d}t + \frac{(1-\alpha)}{1+y^2}\int_{-b}^{+b} q(t)\frac{\mathrm{d}t}{1+t^2} + \frac{2\pi\beta}{1+y^2}q(y) + \frac{\pi(1+\alpha)}{2(1+y^2)}\frac{E_1^*\Delta R}{Q} = 2(1-\beta)\frac{y^2-1}{(y^2+1)^2}. \tag{5}$$

Then, (ii) the problem is normalized over the interval $[-1,+1]$ using the auxiliary variables $\tilde{y} = y/b$ and $\tilde{t} = t/b$, and (iii) the unknown pressure distribution $q(y)$ is expanded in series of even Chebyshev polynomials of the second kind,

$$q(\tilde{y}) = \sum_{i=0}^{N_c} H_i w(\tilde{y}) U_{2i}(\tilde{y}), \quad w(\tilde{y}) = \sqrt{1-\tilde{y}^2}, \tag{6}$$

where $w(\tilde{y})$ is the fundamental function and H_i are $N_c + 1$ unknown coefficients; finally, (iv) by employing a standard collocation procedure (Erdogan et al., 1973; Kaya and Erdogan, 1987) the resulting integral equation is evaluated in $N_c + 1$ collocation points $\tilde{y}_i$

$$\tilde{y}_i = \cos\left[\pi \frac{2i-1}{2N_c+1}\right], \tag{7}$$

leading together with the closing conditions (3), to a system of $N_c + 2$ linear equations to be solved for the $N_c + 1$ coefficients H_i and the load parameter $E_1^* \Delta R/Q$. Therefore, an inverse procedure is used to solve the problem: the contact angle ε is fixed at first, together with material constants α and β, then contact pressure $q(y)$ and load parameter are calculated. A convergence test has shown that the number of collocation points N_c increases as the contact angle is increased, but $N_c = 20$ gives converged results for $\varepsilon \leqslant 120°$.

In order to check the accuracy of the numerical code, a comparison with Persson's results is performed for both identical and dissimilar materials in contact (closed form and numerical results in Persson, respectively), as shown in Tables 1 and 2.

In Fig. 1, the influence of the second Dundurs' parameter β on the maximum contact pressure is depicted by comparing it with the results given for $\beta = 0$. As is shown, the influence of β is practically negligible being less by 2% for every γ, thus the pressure distribution, given by Persson for identical materials, can be used for every couple of mating materials in contact. Notice that there is no symmetry of the error with respect to β, and larger discrepancies are measured for positive values of β. In addition, for γ going to zero, the influence of β also goes to zero as predicted by Hertzian analysis.

Table 1
Comparison with Persson's (closed form) results for identical materials in contact (Persson, 1964, Table 64.1) – $N_c = 20$ at convergence

ε	Persson		Numerical	
	q_0	$E_1^* \Delta R/Q$	q_0	$E_1^* \Delta R/Q$
5.73	6.3794	253.6948	6.3737	253.232
20.41	1.8166	18.7093	1.8164	18.7047
40.61	0.9613	3.7517	0.9613	3.7510
60.23	0.7100	1.0508	0.7099	1.0503
80.06	0.6114	0.1246	0.6114	0.1244

Table 2
Comparison with Persson's (numerical) results for dissimilar materials in contact $\varepsilon = 22.62°$ (Persson, 1964, Table 88.1 and 89.1) – $N_c = 20$ at convergence

α	β	Persson		Numerical	
		q_0	$E_1^* \Delta R/Q$	q_0	$E_1^* \Delta R/Q$
−0.5	−0.175	1.644	31.36	1.639	31.15
−1/3	−0.117	1.645	23.19	1.641	23.06
0	0	1.645	14.98	1.645	14.98
+1/3	+0.117	1.650	10.92	1.649	10.93
+0.5	+0.175	1.652	9.56	1.651	9.58

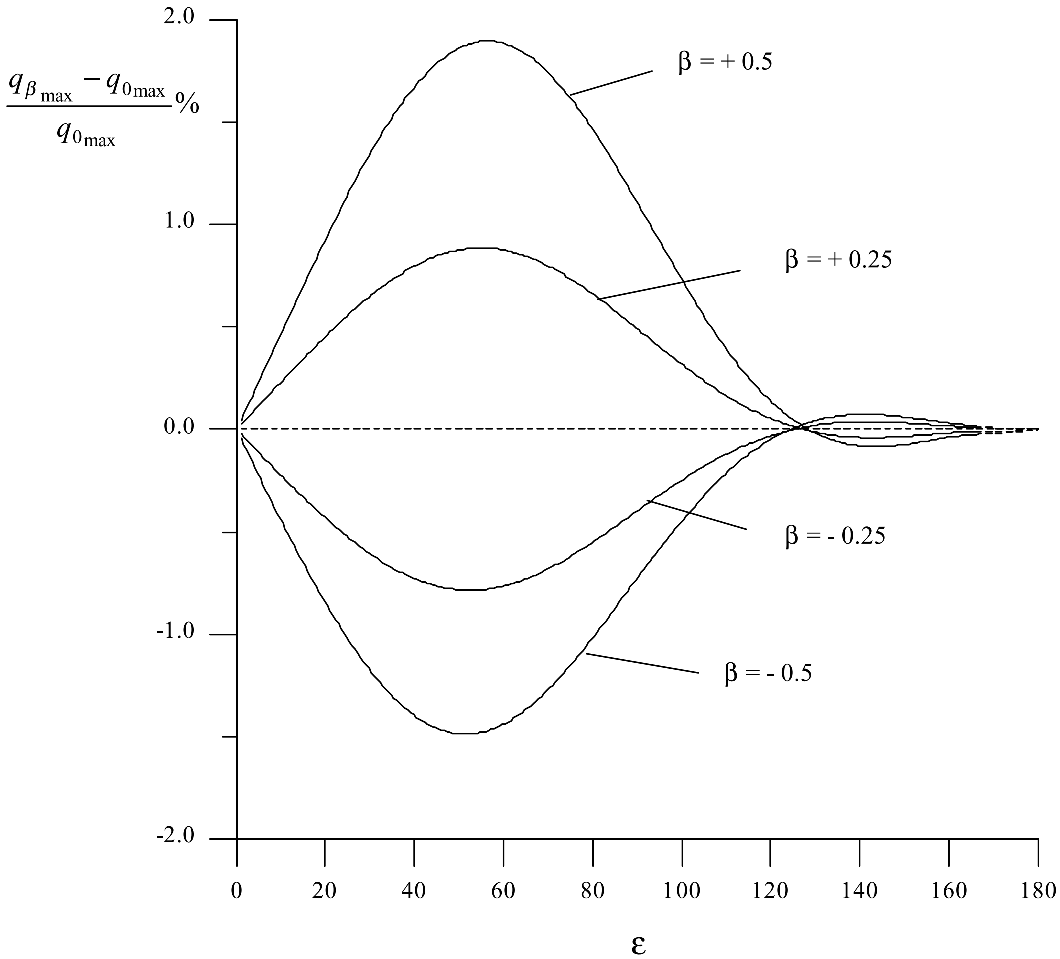

Fig. 1. Effect of second Dundurs' constant (β) on the maximum of the normalized pressure distribution as a function of the contact angle. (q_β is the normalized pressure evaluated for $\beta \neq 0$).

4. Approximate treatment

The previously reported results suggest an approximate treatment for the relationship between the load parameter $E_1^* \Delta R/Q$ and the contact angle ε: since the influence of the second Dundurs' parameter β on the contact pressure is negligible, in the generalized contact angle–load parameter relation (Eq. (2)) the results derived in Part I for elastically similar materials are used for B and $q(y)$. Thus, the pressure distribution is given by

$$q(y) = \frac{2\sqrt{b^2 - y^2}}{\pi\sqrt{b^2+1}(1+y^2)} + \frac{1}{\pi}\left(1 - \frac{B}{2}\right) \log \frac{\sqrt{b^2+1} + \sqrt{b^2-y^2}}{\sqrt{b^2+1} - \sqrt{b^2-y^2}} \tag{8}$$

with

$$B = \frac{2b^4 + 2b^2 - 1}{b^2(b^2 + 1)}. \tag{9}$$

Substituting Eqs. (8) and (9) in Eq. (2) and knowing from Part I, that

$$\int_{-b}^{+b} q(t)\frac{\mathrm{d}t}{1+t^2} = \frac{1}{2\pi}\frac{I_b}{b^2(b^2+1)} + \frac{b^2}{(b^2+1)}, \tag{10}$$

where

$$I_b = -2\pi \log\left[\cos\left(\varepsilon/2\right)\right] = \pi \log\left[1 + b^2\right], \tag{11}$$

we hence obtain the approximate form for the general loading parameter–contact angle relation as

$$\frac{E_1^* \Delta R}{Q} = \frac{(\alpha - 1)(\log\left[b^2 + 1\right] + 2b^4) + 2}{\pi(1+\alpha)(b^2+1)b^2} - \frac{4\beta}{\pi(1+\alpha)}, \tag{12}$$

where the first term was obtained already for $\beta = 0$ in Part I of the paper, and the second term, $-(4\beta/\pi(1+\alpha))$, gives the influence of the second Dundurs' parameter. It is also clear that this influence is greater if the first Dundurs' parameter, α, is closer to -1. In Fig. 2, the influence of the parameter β on the

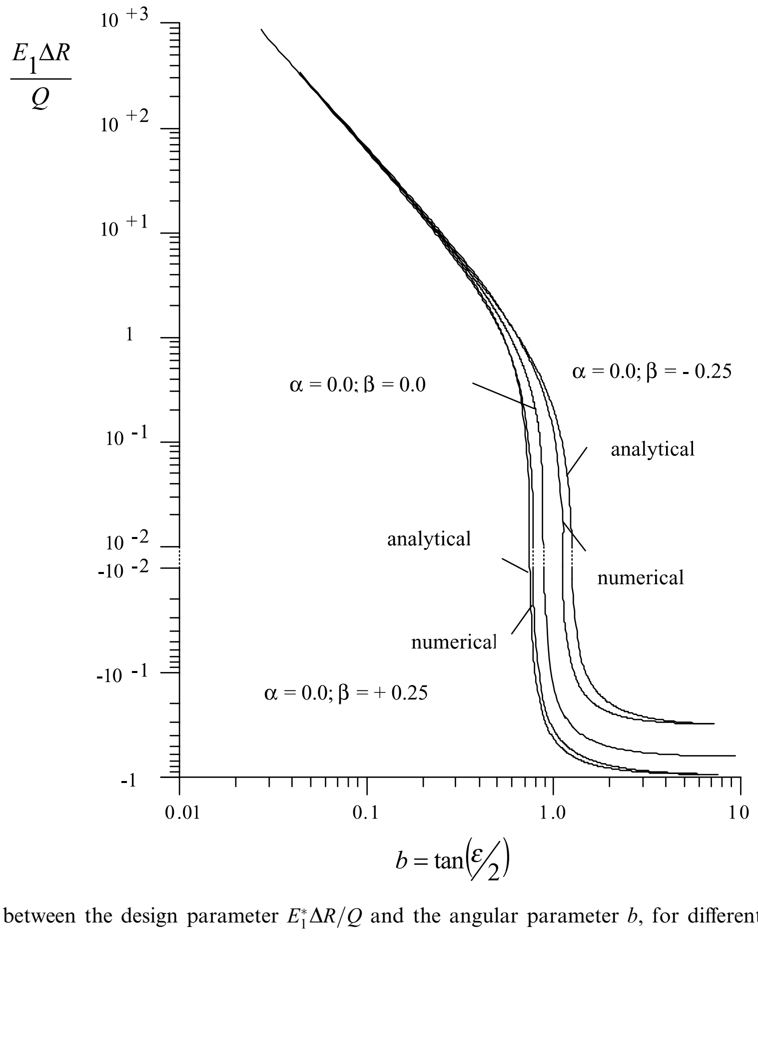

Fig. 2. The relationship between the design parameter $E_1^* \Delta R/Q$ and the angular parameter b, for different values of the Dundurs' parameters $\beta(\alpha = 0)$.

load parameter, for different values of the contact angle, is illustrated for the case of $\alpha = 0$. In addition, a comparison between the numerical solution and the approximate analytical treatment suggested by the authors is also illustrated, showing that results are slightly different only for small load parameters ($E_1^* \Delta R/Q$ approaching zero). In the following table, some values for the limiting contact angle are given together with the associated percentage error.

5. Limiting cases

We now proceed to consider two conditions of special interest, namely (i) the neat-fit or infinite load condition ($E_1^* \Delta R/Q = 0$) and (ii) the complete contact case ($b \to \infty$).

5.1. Neat-fit or infinite load condition

For a zero initial radial clearance ΔR or an infinite applied load Q, the system reaches a limiting condition for which the contact area has a precise extension $\varepsilon_{\lim}$ which depends only on the material parameters α, β. This is of special interest as it defines the border between the advancing contact clearance fit ($\Delta R > 0$) and the receding interference-fit ($\Delta R < 0$). Moreover, under such limiting condition, the problem becomes linear, in that contact pressure, distributed over a constant area, increases linearly with the applied load and so does the severity of the internal stress field.

The limiting angle is evaluated by solving the implicit equation (12), that is

$$\frac{(\alpha - 1)(\log[b^2 + 1] + 2b^4) + 2}{(b^2 + 1)b^2} = 4\beta, \tag{13}$$

where the first term on the right-hand side was obtained already in Part I of the papers (eq. (25)). In Fig. 3, the limiting contact angle is illustrated as a function of the Dundurs' parameters α, β on the Dundurs' plane.

Notice that, even if the numerical and approximate analytical results are slightly different for the neat fit–infinite load condition, as depicted in Fig. 2 and Table 3, Eq. (13) must still be preferred to the numerical treatment. In fact, the numerical procedure is not direct, meaning that the limiting contact angle ($E_1^* \Delta R/Q = 0$) has to be determined iteratively, solving at each step a system of at least 21 equations. However, a comparison with the analytical results given by Noble and Hussain (1969) in Part I, for $\beta = 0$

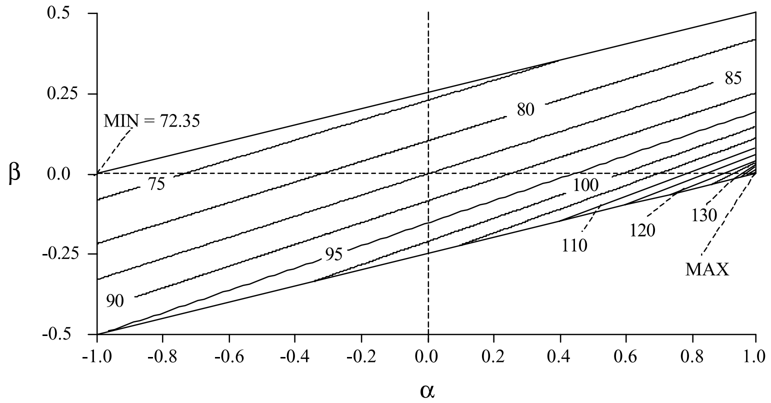

Fig. 3. The limiting contact angle for neat-fit condition, $\varepsilon_{\lim}$, as a function of the Dundurs' parameters α, β.

Table 3
Limiting contact angles ($E_1^* \Delta R/Q = 0$), and comparison between approximate and fully numerical treatment ($N_c = 20$ at convergence)

α	β	Approximate	Numerical	Error (%)
−1.0	−0.5	94.846	87.764	−8
0.0	−0.25	103.899	98.018	+6
0.0	+0.25	74.144	75.725	−2
+1.0	+0.5	76.321	80.962	−6

has shown the great accuracy of the numerical method also in this condition, the error being of less than 0.1% with α ranging from −1 to +1.

5.2. Complete contact

For interference fit condition, the critical load parameter for onset of separation $E_1^* \Delta R/Q$ is found by evaluating the limit of $E_1^* \Delta R/Q$, in Eq. (12), as b tends to infinity, that is

$$\left[\frac{E_1^* \Delta R}{Q}\right]_{\lim} = \frac{2}{\pi}\frac{\alpha - 1 - 2\beta}{\alpha + 1}. \tag{14}$$

This relation coincides with the analytical result given by Ghosh et al. (1981), only for plane stress assumption. In Fig. 4, the negative value of the logarithm of the limiting load is plotted as a function of Dundurs' parameters α, β on Dundurs' plane.

6. Strength of the contact

In Part I of the paper, the strength of the contact has been assessed as a function of the contact parameter b and Poisson's ratio ν_2 (Fig. 9 in Part I). These results can be also employed in the general case of $\beta \neq 0$, providing that the correct contact parameter b and Poisson's ratio ν_2 are used. In fact, at fixed load parameter $E_1^* \Delta R/Q$ and Dundurs' constant α, an increase of β leads to (i) a increase of Poisson's ratio, as shown by the following equation:

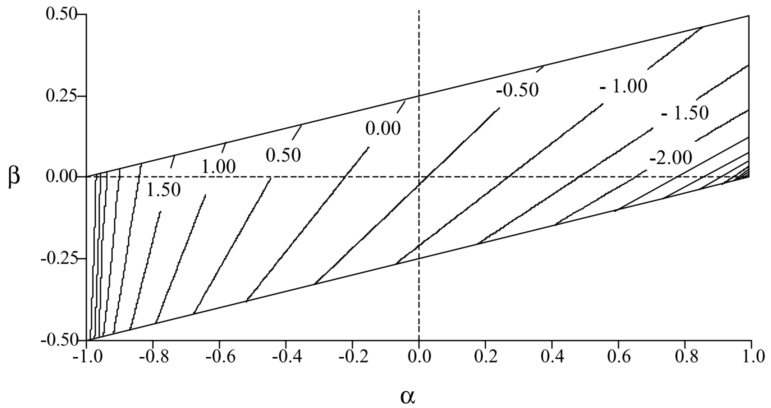

Fig. 4. The negative value of the logarithm of the limiting load, $\left(\frac{Q}{E_1^* \Delta R}\right)_{\lim}$, are function of the Dundur's parameters α, β.

$$\beta = \frac{(1+\alpha)(1-\nu_1) - (1-\alpha)(1-\nu_2)}{4}, \tag{15}$$

which is always beneficial (Fig. 9 in Part I), and (ii) an increase of the contact parameter (Eq. (12)). Since an increase of ν_2 has to be followed by a decrease of the contact parameter b in order to maximize the admissible load, generally, an increase of β is (i) always beneficial if the initial contact angle ($\beta = 0$) is sufficiently small and (ii) could be also beneficial if the system is initially ($\beta = 0$) contact angle is very close to the maximum.

7. Computational algorithm

The formulas given so far, in Parts I and II of the paper, can be implemented in symbolic mathematical softwares in order to determine efficiently semi-angle of contact ε, dimensionless pressure distribution $q(y)$, stress field and yielding parameter for different material properties (α and β) and load parameters ($E_1^* \Delta R/Q$) of the pinned connection. In the following diagram, a general computational algorithm is presented for both elastically similar and elastically dissimilar materials.

$E_1^* \Delta R/Q;\ \alpha;\ \beta$	Elastically similar materials, Part I	Elastically dissimilar materials
$\Downarrow$	Eq. (25)	Part II, Eq. (12)
ε		
$\Downarrow$	Eq. (8)	Part II, Eq. (8)
$q(y)$		
$\Downarrow$	Eq. (37)	Part I, Eq. (37)
$\tilde{\sigma}_{rr};\ \tilde{\sigma}_{\phi\phi};\ \tilde{\sigma}_{r\phi}$		
$\Downarrow$	Eq. (41)	Part I, Eq. (41)
$\left[\frac{\sqrt{J_2}}{Q/R_2}\right]_{\max}$		

It is straightforward to modify the present algorithm if the problem has to be solved for the load parameter having fixed ad priori the semi-angle of contact.

8. Conclusions

A completely closed form solution (although with a minor approximation in the general elastic dissimilarity case) has been given to the problem of cylindrical bodies in contact. The effect of both α and β is strong on the relation load to contact area-size, and in particular on the limiting angle of contact, and on the angle of limiting complete contact condition for an interference contact.

The pressure distribution depends *only* on the angle of contact (in the spirit of the approximation), which in turn depends on the loading conditions (the parameter $E_1^* \Delta R/Q$, below) and both α and β. In other words, once the contact area has been determined from this general relation, all the results from the Part I of the paper can be used in terms of strength of the contact. A complete assessment of this problem is therefore achieved, and numerical results confirm the approximation of the closed form results is indeed very good.

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